International Journal of Theoretical Physics, Vol. 43, No. 9, September 2004 (© 2004)

Superposition, Entropy and Schmidt Decomposition of States

Antonio Zecca

Superposition and entropy are compared using the language of the logic of quantum mechanics. It is pointed out that a finite value of the relative quantum entropy of states implies a superposition relation between the states themselves. The superposition relation is then studied by comparing the pure state of the compound system with the product of the reduced states and an intermediate "Schmidt" state. All the corresponding relative quantum entropies are evaluated in terms of the Schmidt coefficients of the global pure state. Some of the results are extended in case the compound system is in a state represented by a general density operator.

KEY WORDS: superposition; von Neumann entropy; Schmidt decomposition.

1. INTRODUCTION

Entropy considerations for quantum systems had been made possible by enlarging the notion of state to include density operators. These states, introduced by von Neumann in 1920s (e.g. von Neumann, 1955) are the quantum counterpart of the classical statistical mixtures of states. The quantum entropy and the relative entropy functions have been widely studied in the literature having applications in many fields of physics. Mathematical aspects have been put into evidence in the particular case of compound system where the entropy of the system and the entropies of the subsystems are related by important well-known inequalities. (For a review on the argument see Wehrl (1948) and Ruskai (2002); see also Fan (2003).) On account of the fact that the classical entropy, originally introduced by Shannon (1948), has an expression similar to the quantum one, the role of entropy has been recently reconsidered. The downfall of the general results on the entropy functions intersects now with many fields of research such as quantum information and quantum computing (Nielsen and Chuang, 2000), quantum communication (Colin, 1999; Janzing and Beth, 2003); quantum teleportation

[1] Dipartimento di Fisica dell' Universita' and INFN, Via Celoria 16, I-20133 Milano, Italy
[2] GNFM, Milano, Italy; e-mail: Zecca@mi.infn.it

(Bennet *et al.*, 1985; Bowen and Bose, 2001; Ban, 2003) and also classical informations and black holes (Hosoya and Carlini, 2002). In all these fields an important role is played by the notion of entanglement and Schmidt decomposition of states (e.g. Peres, 1995). Owing to the fact that the superposition relation can as well be formulated for the density operators (Varadarajan, 1968; Zecca, 1980), it seems natural to study the mutual dependence of entropy and superposition in situations possibly involving entangled states. That problem is discussed here in the language of the logic approach to quantum mechanics. Accordingly, some properties of superposition under tensor product and partial trace operations are primarily discussed. It is pointed out that finite values of the quantum relative entropy is meaningful only for states that are in an explicit superposition relation. The study is first developed in case the compound system is in a pure state. The reduced states, their product state and an intermediate "Schmidt-like" state between them are compared both for what concerns superposition as well as for the evaluation of their relative entropies. All the results are expressed in terms of the Schmidt coefficients of the global pure state. The method is then generalized in an elementary way to the case of a generic initial state of the compound system.

2. SUPERPOSITION AND ENTROPY IN STANDARD LOGIC: DEFINITIONS AND PRELIMINARY RESULTS

The language of the standard logic approach to quantum mechanics is useful for the present treatment. (The approach had its origin from a paper of Birkhoff and von Neumann (1936); for further developments see Jauch (1968); Piron (1970); Beltrametti and Cassinelli (1981).) Accordingly, to the physical system there is an associated pair $L,\ S$. The logic $L \equiv L(H)$ is the complete orthomodular atomic lattice of the closed subspaces (propositions) of a separable complex Hilbert space H of dimension ≥ 3. (The operations in $L(H)$ are such that $\vee_\alpha a_\alpha,\ a_\alpha \in L(H)$ denotes the closed subspace generated by the subspaces a_α of H, while $\wedge_\alpha a_\alpha \equiv \cap_\alpha a_\alpha$.) The set of the states S is the set of the σ-additive probability measures on $L(H)$. The probability of the outcome yes for a test of proposition a when the system has been prepared according to the procedures of the state s is denoted by $s(a)$. By Gleason's theorem (Gleason, 1957) there is an affine isomorphism between S and the set $K(H)$ of the positive trace class operators of H of trace 1 (density or statistical operators). Therefore, for every s there is one and only one $\rho \in K(H)$ so that

$$s(a) \equiv \rho(a) = \mathrm{Tr}\rho P^a, \quad (a \in L(H)) \tag{1}$$

where P^a is the orthogonal projection in H of range a. In case a is one dimensional we write $a \equiv [\psi],\ \ P^a = P^\psi$, ψ being a unit vector of a.

Definition 2.1. A state ρ is said to be superposition of the states of $D \subset K(H)$ if anyone of the following equivalent conditions hold:

$$a \in L(H), \quad \sigma(a) = 0 \quad \forall\, \sigma \in D \quad \Rightarrow \quad \rho(a) = 0 \tag{2}$$

$$b \in L(H), \quad \sigma(b) = 1 \quad \forall\, \sigma \in D \quad \Rightarrow \quad \rho(b) = 1 \tag{3}$$

Formulations (2) and (3) are indeed equivalent as it is easily seen. Definition (2) was proposed by Varadarajan (1968) for a general quantum logic. Its application to the standard logic has been studied in Zecca (1980). It is useful to recall that anyone of the equivalent conditions (2) and (3) is in turn equivalent to the condition

$$[\rho] \leq \bigvee_{\sigma \in D} [\sigma] \tag{4}$$

where $[\rho]$ denotes the range of ρ as an operator in H, as it can be shown by considering the spectral decomposition of the density operators (Gorini and Zecca, 1975; Zecca, 1980). The definition includes the pure superposition of pure states $\psi = \alpha\psi_1 + \beta\psi_2$, $(|\alpha|^2 + |\beta|^2 = 1)$ for which obviously $[\psi] \leq [\psi_1] \vee [\psi_2]$ as well as the statistical mixtures of states $\rho = \sum_i \alpha_i \rho_i$ for which $[\rho] = \vee_i [\rho_i]$. (In case of the normal states of a $W^\star$ algebra the relation can be expressed by $\operatorname{supp} \rho \leq \bigvee_{\sigma \in D} \operatorname{supp} \sigma$ (Zecca, 1981).)

In dealing with physical system of Hilbert space H that is compound of two subsystems with Hilbert spaces H_1 and H_2 one generally assumes $H = H_1 \otimes H_2$. It is then possible to obtain reduced states, starting from a state of the compound system, by taking partial traces. If $\{u_{1h}\}$ and $\{v_{2h}\}$ are complete ortho-normal systems in H_1 and H_2, respectively, and $\rho \in K(H)$ we denote $\rho_1 = \mathrm{Tr}_2 \rho = \sum_h < v_{2h}|\rho|v_{2h}\rangle$, $\rho_2 = \mathrm{Tr}_1 \rho = \sum_k \langle u_{1k}|\rho|u_{1k}\rangle$ as the reduced states. It is a fact that the superposition relation is invariant both under tensor product as well as under partial trace operations (Zecca, 2003). This means, by using result (4), that if $\rho \in K(H)$, $\rho^\alpha \in K(H_1)$ and $\sigma \in K(H_2)$, $\sigma^\alpha \in K(H_2)$, then the following conditions hold:

$$[\rho] \leq \bigvee_\alpha [\rho^\alpha], \quad [\sigma] \leq \bigvee_\beta [\sigma^\beta] \qquad \Longleftrightarrow \qquad [\rho \otimes \sigma] \leq \bigvee_{\alpha,\beta} [\rho^\alpha \otimes \sigma^\beta] \tag{5}$$

$$[\rho] \leq \bigvee_\alpha [\rho^\alpha] \quad \Longrightarrow \quad [\rho_1] \leq \bigvee_\alpha \left[\rho_1^\alpha\right] \quad \text{and} \quad [\rho_2] \leq \bigvee_\alpha \left[\rho_2^\alpha\right]. \tag{6}$$

The object is now of briefly pointing out some aspects of the definition of the entropy functions from the point of view of the superposition relation.

Definition 2.2. The quantum entropy of the state $\rho \in K(H)$ and the relative quantum entropy of the states ρ, σ are given by $S(\rho) = -\mathrm{Tr}\rho \log \rho$, and $S(\rho|\sigma) = -\mathrm{Tr}\rho(\log \rho - \log \sigma)$, respectively.

By considering the spectral decomposition $\rho = \sum_i \rho_i P^{\psi_i}$ (e.g. Schatten, 1960) one has immediately that $S(\rho) = -\sum \rho_i \log \rho_i \geq 0$ and equality holds if and only if ρ is a pure state P^{ψ}. Also $S(\rho|\sigma) \geq 0$ (with equality holding if and only if $\rho = \sigma$) as a consequence of Klein's inequality. (A list of properties and inequalities to which the entropy functions satisfy as well as their mutual interdependence are widely considered in the review by Ruskai (2002); see also Fan (2003).) Here we are interested only in some remarks on the definitions and, in the next section, in the study of some elementary situations.

Consider then the states $\rho, \sigma \in K(H)$, the spectral decomposition $\sigma = \sum_i \sigma_i P^{\phi_i}$ and the complete ortho-normal set of states $\{\tilde{\phi}_l\} = \{\phi_h\} \cup \{\phi'_l\} \cup \{\phi''_j\}$ in H where $\{\phi_h\} \cup \{\phi'_l\}$ generates $[\sigma] \vee [\rho]$. (The set $\{\phi'_l\}$ is possibly empty.) Then the relative entropy takes the form

$$S(\rho|\sigma) = -S(\rho) - \sum_l \log \sigma_l \langle \phi_l|\rho|\phi_l\rangle - \sum_i \langle \phi'_i|\rho \log \sigma|\phi'_i\rangle \tag{7}$$

If $\{\phi'_i\}$ is non empty, $\sigma|\phi'_i\rangle = 0|\phi'_i\rangle$ so that $S(\rho|\sigma) = \infty$. Therefore, if $S(\rho) < \infty$, $S(\rho|\sigma)$ takes finite values if and only if $[\rho] \leq [\sigma]$, that is if and only if ρ is superposition of σ. (Alternatively, if and only if $[\rho]^{\perp} \geq [\sigma]^{\perp}$ or $ker\ \rho \geq ker\ \sigma$ (e.g. Ruskai, 2002) or, in the language of the normal states of a $W^\star$ algebra, if and only if supp $\rho \leq$ supp σ (e.g. Zecca, 1981).)

As an application, consider two states such that $[\rho] = [\sigma]$, with $\dim[\rho] = N < \infty$, whose spectral decompositions are of the form $\rho = \sum_{i=1}^N \rho_i P^{\phi_i}$, $\gamma = \sum_{i=1}^N \gamma_i P^{\phi_i}$ with $\rho_i, \gamma_i > 0$, $\sum_{i=1}^N \rho_i = \sum_{i=1}^N \gamma_i = 1$. Since ρ, γ are superposition of each other, both $S(\rho|\gamma) \geq 0$ and $S(\gamma|\rho) \geq 0$ or

$$\sum_{i=1}^N \gamma_i \log \frac{\gamma_i}{\rho_i} \geq 0, \qquad \sum_{i=1}^N \rho_i \log \frac{\rho_i}{\gamma_i} \geq 0. \tag{8}$$

By choosing $\gamma_i = 1/N$, $i = 1, 2, \ldots, N$ one gets

$$-\sum_{i=1}^N \rho_i \log \rho_i \leq \log N, \qquad \prod_{i=1}^N \rho_i \leq \frac{1}{N^N}. \tag{9}$$

Therefore, as is well known, $S(\rho)$ takes its maximum for $\rho_i = 1/N$ for every i. Relation (9) can be extended to hold for $N \to \infty$ in which case, since $\rho_i \to 0$, it must be $\prod_i^N \rho_i = 0$.

3. PARTICULAR SITUATIONS

Some considerations can be developed in case the compound system is in a pure state. Many results can then be expressed entirely in terms of the Schmidt coefficients. Suppose indeed $\phi \in H_1 \otimes H_2$ and consider its Schmidt decomposition

(Schmidt, 1906)

$$\phi = \sum_i g_i |u_{1i} \otimes v_{1i} > \quad (g_i > 0) \tag{10}$$

where $\langle u_{1i}|u_{1k}\rangle_1 = \delta_{ik}$, $\langle v_{2j}|v_{2l}\rangle_2 = \delta_{jl}$, $\sum_i g_i^2 = 1$. One can derive the one-dimensional projection $P^\phi = |\phi\rangle\langle\phi|$, the reduced states P_1^ϕ, P_2^ϕ and the product state $P_1^\phi \otimes P_2^\phi$. Explicitly (compare with Ekert and Knight, 1995) one obtains:

$$P^\phi = \sum_{ik} g_i g_k |u_{1i}\rangle\langle u_{1k}| \otimes |v_{2i}\rangle\langle v_{2k}| \tag{11}$$

$$P_1^\phi = \sum_i g_i^2 |u_{1i}\rangle\langle u_{1i}| \tag{12}$$

$$P_2^\phi = \sum_k g_k^2 |v_{2k}\rangle\langle v_{2k}| \tag{13}$$

$$P_1^\phi \otimes P_2^\phi = \sum_{ik} g_k^2 g_i^2 |u_{1i}\rangle\langle u_{1i}| \otimes |v_{2k}\rangle\langle u_{2k}| \tag{14}$$

One can also consider the “Schmidt-like” state

$$\rho_\phi(f) = \sum_i f_i \, |u_{1i}\rangle\langle u_{1i}| \otimes |v_{2i}\rangle\langle u_{2i}| \tag{15}$$

(for any countable set of positive numbers f_i with $\sum_i f_i = 1$) whose expression coincides, as it happens also for P_1^ϕ, P_2^ϕ, with its own spectral decomposition.

Proposition 3.1. *The state P^ϕ is superposition of the state $\rho_\phi(f)$ that in turn is superposition of the state $P_1^\phi \otimes P_2^\phi$.*

Proof: Denote by $\{\tilde{u}_\alpha\}$ $(\{\tilde{v}_\beta\})$ a complete system of vectors in H_1 (H_2) that contains $\{u_{1i}\}$ $(\{v_{2k}\})$. Suppose $\mathrm{Tr}\, P^a \rho_\phi(f) = 0$ and use $\{\tilde{u}_\alpha \otimes \tilde{v}_\beta\}$ to calculate the trace toghether with expression (15). This finally leads to $P^a|u_{1k} \otimes v_{2k}\rangle = 0$ for every k. On the other hand one obtains $\mathrm{Tr} P^a \, P^\phi = \sum_{ik} g_i \, g_k \, \langle u_{1k} \otimes v_{2k}|P^a|u_{1i} \otimes v_{2i}\rangle$ that vanishes by the last result. This proves $[P^\phi] \leq [\rho_\phi(f)]$. The proof of $[\rho_\phi(f)] \leq [P_1^\phi \otimes P_2^\phi]$ is immediate from the very definition of the states. □

It can be checked that $\rho_\phi(f)$ in (15) is not only a disentangled state in the sense of Rudolph (2001), but also it is operator Schmidt decomposed in the sense of Tyson (2003a,b). According to *Proposition* 1 and result (4), $\rho_\phi(f)$ represents a refinement of $P_1^\phi \otimes P_2^\phi$ for what concerns the superposition relation in the sense that

$$[P^\phi] \leq [\rho_\phi(f)] \leq \left[P_1^\phi \otimes P_2^\phi\right]. \tag{16}$$

Relation (16) enables, according to the previous remarks, the calculation of the relative quantum entropies in term of the Schmidt coefficients of ϕ. By remarking that

$$\rho_\phi(f)|u_{1\alpha} \otimes v_{2\beta}\rangle = \delta_{\alpha\beta}\, f_\alpha |u_{1\alpha} \otimes v_{2\beta}\rangle \tag{17}$$

$$P^\phi |u_{1\alpha} \otimes v_{2\beta}\rangle = \delta_{\alpha\beta} \sum_i g_i g_\alpha |u_{1i} \otimes v_{2i}\rangle \tag{18}$$

$$P_1^\phi \otimes P_2^\phi |u_{1\alpha} \otimes v_{2\beta}\rangle = g_\alpha^2 g_\beta^2 |u_{1\alpha} \otimes v_{2\beta}\rangle \tag{19}$$

the following expressions can be easily obtained:

$$S(P^\phi|\rho_\phi(f)) = -\mathrm{Tr}\, P^\phi \log \rho_\phi(f) = -\sum_k g_k^2 \log\, f_k \tag{20}$$

$$S\big(\rho_\phi(f)|P_1^\phi \otimes P_2^\phi\big) = \sum_k f_k \log\big(f_k/g_k^4\big) \tag{21}$$

$$S\big(P^\phi|P_1^\phi \otimes P_2^\phi\big) = -\sum_l g_l^2 \log\, g_l^4 \tag{22}$$

Therefore, by choosing $f_k = g_k^2$ for every k, one has also

$$-\sum_i g_i^2 \log\, g_i^2 = S\big(P_1^\phi\big) = S\big(P_2^\phi\big) = S(P^\phi|\rho_\phi(f)) = S\big(\rho_\phi(f)|P_1^\phi \otimes P_2^\phi\big)$$
$$= \frac{1}{2} S\big(P^\phi|P_1^\phi \otimes P_2^\phi\big) \tag{23}$$

Some of the previous considerations can be extended in case the global system is in a state represented by a density operator.

Proposition 3.2. *Let $\rho \in K(H_1 \otimes H_2)$ with spectral decomposition $\rho = \sum_i \rho_i P^{\psi_i}$ and reduced states $\rho_1 = \sum_j \rho_j P_1^{\psi_j}$, $\rho_2 = \sum_k \rho_k P_2^{\psi_k}$ and define $\rho(f) = \sum_j f_j P_1^{\psi_j} \otimes P_2^{\psi_j}$ for every set of positive numbers f_j such that $\sum_j f_j = 1$. Then ρ is superposition of $\rho(f)$ which in turn is superposition of $\rho_1 \otimes \rho_2$.*

Proof: By assumption and by applying result (6) one has $[\rho_1] = \bigvee_i [P_1^{\psi_i}]$ and similarly for ρ_2. The previous result (16) gives now $[P^{\psi_i}] \leq [P_1^{\psi_i}] \otimes [P_2^{\psi_i}] = [P_1^{\psi_i} \otimes P_2^{\psi_i}]$ for every i (compare with Zecca, 1994). Therefore

$$\bigvee_i [P^{\psi_i}] \leq \bigvee_i \big[P_1^{\psi_i}\big] \otimes \big[P_2^{\psi_i}\big] \leq \left(\bigvee_i \big[P_1^{\psi_i}\big]\right) \otimes \left(\bigvee_k \big[P_2^{\psi_k}\big]\right)$$

and there follows $[\rho] \leq [\rho(f)] \leq [\rho_1] \otimes [\rho_2] = [\rho_1 \otimes \rho_2]$. □

It must be noted here that the expression of $\rho(f)$ has a separable (disentangled) form in the sense of Rudolph (2001), but it has not an operator-Schmidt decomposition in the sense of Tyson (2003a,b), because in general, Tr $P_1^{\psi_i} P_1^{\psi_k} \neq 0$ for $i \neq k$.

REFERENCES

Ban, M. (2003). Quantum teleportation and quantum dense coding in a finite-dimensional Hilbert space. *International Journal of Theoretical Physics* **42**, 1.

Beltrametti, E. and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*. Addison Wesley, Reading, Massachusetts.

Bennet, C. H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., and Wootters, W. K. (1993). Teleporting an unknown quantum state via dual classical and Einstein-Podolski-Rosen channels. *Physical Review Letters* **70**, 1895.

Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics* **37**, 823.

Bowen, G. and Bose, S. (2001). Teleportation as a depolarizing quantum channel, relative entropy and classical capacity. *Physical Review Letters* **87**, 267901.

Colin, P. W. (Ed.) (1999). *Quantum Computing and Quantum Communication*. Springer Lecture Notes in Computer Sciences, Berlin.

Ekert, A. and Knight, P. (1995). Entangled quantum systems and the Schmidt decomposition. *American Journal of Physics* **63**, 415.

Fan, H. (2003). A note on quantum entropy inequalities and channel capacities. *Journal of Physics A* **36**, 12081.

Gleason, A. M. (1957). Measures on the closed subspaces of Hilbert space. *Journal of Mathematics and Mechanics* **6**, 885.

Gorini, V. and Zecca, A. (1975). Reversible dynamics in a proposition-state structure. *Journal of Mathematical Physics* **16**, 667.

Hosoya, A. and Carlini, A. (2002). Quantum entropy bound by information in black hole space-time. *Physical Review D* **66**, 104011.

Janzing, D. and Beth, T. (2003). *Synchronizing quantum clock with classical one-way communications: Bounds on the general entropy*. quan-ph/0306023.

Jauch, J. M. (1968). *Foundation of Quantum Mechanics*. Addison Wesley, Reading, Massachusetts.

Nielsen, M. A. and Chuang, I. L. (2000). *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge.

Peres, A. (1995). *Quantum Theory: Concept and Methods*. Kluwer Academic Publishers, Dordrecht.

Piron, C. (1976). *Foundation of Quantum Physics*. Benjamin, Reading, Massachusetts.

Rudolph, O. (2001). A new class of entanglement measures. *Journal of Mathematical Physics* **423**, 5306.

Ruskai, M. B. (2002). Inequalities for quantum entropy: A review with conditions for equality. *Journal of Mathematical Physics* **43**, 4358.

Schatten, R. (1960). *Norm Ideals of Completely Continous Operators*. Springer Verlag, Berlin.

Schmidt, E. (1906). Zur theorie der linearen und multilinearen integralgleichungen. *Methematische Annalen* **63**, 433.

Shannon, C. (1948). The mathematical theory of communication. *Bell System Technical Journal* **27**, 379; **27**, 623.

Tyson, J. (2003a). Operator-Schmidt decomposition of the quantum Fourier transform on $C^{N_1} \bigotimes C^{N_2}$. *Journal of Physics A* **36**, 6813.

Tyson, J. (2003b). Operator-Schmidt decomposition and the Fourier transform, with applications to the operator-Schmidt numbers of unitaries. *Journal of Physics A* **36**, 10101.

Varadarajan, V. S. (1968). *The Geometry of Quantum Theory*. D. Van Nostrand Company, Princeton, New Jersey.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, Princeton, NJ.

Wehrl, A. (1978). General properties of entropy. *Review of Modern Physics* **50**, 221.

Zecca, A. (1980). Dirac's superposition of pure states extended to the statistical operators. *International Journal of Theoretical Physics* **19**, 629.

Zecca, A. (1981). The superposition of the states and the logic approach to quantum mechanics. *International Journal of Theoretical Physics* **29**, 191.

Zecca, A. (1994). Product of proposition-state structures preserving superposition. *International Journal of Theoretical Physics* **33**, 983.

Zecca, A. (2003). Superposition, entanglement, and product of states. *International Journal of Theoretical Physics* **42**, 1621.